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Anisotropic Transport of Charge and Complexified Duality

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Charge transport in two dimensions provides an ideal laboratory for investigating parameter space geometries. The Onsager relations for anisotropic transport in a parity-violating external field endow these spaces with a highly nontrivial complex (and Kähler) structure, which can be given a simple geometrical interpretation. A large class of Coulomb gases exhibiting this structure have a generalized Kramers–Wannier symmetry (complexfield duality) which is contained in the modular group. Knowledge of this symmetry and the degrees of freedom encoded in the Coulomb gas appear to be sufficient to determine the global phase diagram and the renormalization group fixed-point structure, including the critical exponents. This accounts for all the scaling behavior observed so far in the quantum Hall system.

KEY WORDS: Anisotropic transport; charge; duality.

The quantum Hall effect[1] is the prototype example of the novel and interesting physics which can take place when charge carriers are effectively confined to two dimensions. The most exciting possibility is if these turn out to be anyons, particles intermediate between fermions and bosons,[2] since that would mean that we have opened up a vast new category of quantum phenomena to experimental and theoretical investigation. Both the equilibrium and nonequilibrium (transport) properties of anyonic systems are exotic and at present poorly understood, even in the simple quantum Hall system. It is therefore of paramount importance to elucidate the structure of the effective quantum field theory describing the possible ground states and excitations of this system.

So far work on this problem has been restricted to the isotropic case, i.e., to samples whose composition (on scales much larger than the lattice

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spacing) is homogeneous and rotationally invariant in the plane to which transport is confined. That discussion should be expanded to include the anisotropic case, not only because anisotropic samples are now becoming available, but also because they provide valuable information about the structure of the effective field theory. The easiest way to see that this is the case is by adopting the "phenomenological" approach proposed in ref. 3.

Briefly, the main idea in ref. 3 is that already available scaling data[4,5] suggest that an infinite discrete group is acting on the space of transport coefficients, i.e., the conductivities or resistivities. Scaling signals critical behavior and therefore gives *local* information about the phase structure and renormalization group (RG) flow on the space of coupling constants (RG parameters). Thus, the observed scaling in the magnetoresistance tensor at special points in the parameter space, identified as critical points at which the charge-carrying states delocalize (thus allowing the conductivities to change), provides powerful constraints on any theory which aspires to account for the quantum Hall effect. Furthermore, since the value of the scaling exponent appears to be the same for *all* transitions, i.e., between any pair of neighboring plateaus, this also sheds light on the *global* structure of the phase and flow diagram. The suggestion is that the mere existence of such a "superuniversal" scaling exponent for completely distinct phase transitions indicates that there must be a discrete Kramers–Wannier-like symmetry connecting the different fixed points. Such a symmetry partitions the parameter space $H=(\sigma_{xy}, \sigma_{xx}>0)$ into universality classes labeled by the Hall fractions to which they are attached, and also automatically enforces the observed "superuniversality" of the delocalization exponent. Keeping also in mind that infrared stable fixed points, i.e., attractors of the RG flow, must only appear at fractions (Hall plateaus) $(p/q, 0)$, the supply of candidate groups is extremely limited. The simplest choice (this is the phenomenological ansatz), which can account for any type of fraction, is the modular group. The resulting global phase and flow diagram is consistent with all available scaling data.

In ref. 6 it was argued that these symmetries have a simple physical interpretation in a localization theory of anyons. In dirty two-dimensional samples we expect almost all charge carriers to be trapped in the potential wells provided by the impurities, for almost all values of the external control parameters. Typically there is therefore no charge transport. However, for exceptional values of the external magnetic field it may happen that the localized wave functions spread out just enough to "percolate" through the sample, thus allowing charge to slip through, with a corresponding change in the transport coefficients. This is a critical phenomenon called the delocalization transition. It is the universal character of this percolation transition which is encoded in the discrete global symmetry. The delocalization exponent is $\nu = 4/3$, or close to 7/3 if tunneling corrections should be taken into account.

These results on the location and scaling properties of the delocalization fixed points are in agreement with numerical work[7–9] and experiment,[4,5] as well as more recent theoretical considerations based on a mean-field anyon picture.[10]

This successful global approach to the quantum Hall system can be extended to anisotropic magnetotransport, which recently has come within experimental reach.[11] A sample with a two-dimensional electron gas with in-plane anisotropy has been made by Stormer *et al.*[12] The structure was formed by modulation-doped molecular-beam epitaxy overgrowth on the cleaved edge of an AlGaAs compositional superlattice. Low-temperature magnetotransport measurements revealed clear quantum Hall characteristics. The electron mobility and density in this anisotropic sample are similar to the mobility and density in the isotropic sample used in the Princeton experiment[4] to determine the fixed-point structure for the isotropic case, so a determination of the fixed-point characteristics of such anisotropic samples should be possible.

The virtue of generalizing this discussion to the anisotropic case is twofold:

(i) Comparison with experiment. By providing precise predictions it expands the range of experiments which should be performed in order to gather evidence for or against the scaling hypotheses mentioned above.

(ii) Comparison with theory. It provides theoretical (geometrical) constraints on the structure of the effective quantum field theory which presumably is responsible for the stunning and unusual scaling behavior observed in the quantum Hall system.

The value of (i) needs no further comment, and while the formulation of the generalization is simple in the language of (differential) geometry (tensors), it is not obvious. The value, and especially the implementation, of (ii) may be less obvious. It is clear that any theory aspiring to account for all universal properties of the quantum Hall system must include anisotropy in a natural and elegant way. This will be the case if the coupling constants of the effective field theory transform as a rank-two tensor which is simply related to the transport coefficients. Due to the presence of the antisymmetric Hall conductivity the geometrical structure of such a theory cannot be conventional (for instance, no ordinary sigma model will suffice). The additional requirement (ansatz) of modular parameter space symmetry (motivated by its success in the isotropic case), which is similar to but distinct from the parameter space transformations induced by the coordinate transformations discussed below, further complicates the construction of such an effective theory. These symmetries are disentangled below in order to clarify the geometrical structure which we believe must be encoded in the effective field theory of the quantum Hall system.

The virtue of the "phenomenological" procedure should now be clear: if, as appears to be the case for the isotropic data mentioned above, the anisotropic scaling data confirm the predictions made here, then we will have a very strong constraint on the underlying field theory. It should reduce at large scales, and in the static limit, to an effective model invariant under an infinite discrete group acting on the parameter space, and that is presumably not a property of very many theories. It is, for example, not at all obvious that a localization theory of anyons (which presumably can be encoded in some kind of Chern–Simons field theory) will have this property.[6]

To see how we are led to a particular class of models from a macroscopic point of view, consider first the isotropic case. The Onsager relations and rotational symmetry of the isotropic sample force the conductivity tensor to be a complex number[2]:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} = \sigma_{xx} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sigma_{xy} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma_{xx} + i\sigma_{xy} \tag{1}$$

because the antisymmetric two-tensor squares to -1. The third law of thermodynamics forces σ_{xx} to be positive. Hence the parameter space in this case is a complex half-plane, and it is convenient to choose the upper half-plane: $H(\sigma) = \{\sigma = \sigma_{xy} + i\sigma_{xx}, \sigma_{xx} > 0\}$, since this is where the modular group and its siblings conventionally act by fractional linear transformations. With this choice of parametrization the resistivity $\rho = \rho_{xy} + i\rho_{xx}$ is simply $\rho = S(\sigma) = -1/\sigma$.

It seems obvious that any theory of the isotropic quantum Hall system, phenomenological or more fundamental, should immediately generalize to the anisotropic case in a simple and natural manner, since there is nothing sacrosanct about isotropic transport. In the anisotropic case we have four independent components of the transport tensor, but since any off-diagonal part of the symmetric (dissipative) transport tensor can be eliminated by rotating the coordinate system so that it is aligned with the currents, we can without loss of generality consider an anisotropic transport matrix of the form

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix} + \sigma_{xy} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

[2] I am grateful to J. Myrheim for this remark.

The dissipative (symmetric) part of the transport matrix can be reduced to the unit matrix by rescaling the coordinates:

$$x \to \eta^{1/2} x, \qquad y \to \eta^{-1/2} y \tag{3}$$

by the *anisotropy parameter*

$$\eta \equiv \left(\frac{\sigma_{yy}}{\sigma_{xx}}\right)^{1/2} \tag{4}$$

so that

$$\sigma_{ij}^{\text{diss}} = (\sigma_{xx}\sigma_{yy})^{1/2}\,\delta_{ij} \tag{5}$$

Because the antisymmetric two-tensor squares to -1, we can therefore always represent the physically interesting part of the anisotropic transport tensor by the complex number:

$$\sigma = \sigma_{xy} + i(\sigma_{xx}\sigma_{yy})^{1/2} \tag{6}$$

The eigenvalues, i.e., two dissipative conductivities σ_{xx} and σ_{yy}, must again be positive by the third law. Hence the parameter space in this case is a complex half-plane. With this choice of parametrization the resistivity $\rho = \rho_{xy} + i(\rho_{xx}\rho_{yy})^{1/2}$ is simply $\rho = S(\sigma) = -1/\sigma$. The phenomenological ansatz is that the modular group acts *linearly* on this natural complex structure. The resulting fixed-point structure is in agreement with numerical work[7-9] and scaling experiments[4,5] on isotropic systems.

This anisotropic generalization has a natural and unique geometrical interpretation. Observe first that (1) provides the two-dimensional parameter space H with a natural complex structure, which suggests that we ask: *what natural geometrical structure is associated with the four-dimensional parameter space of the anisotropic system?*

In order to answer this question we recall first that the geometry of a parameter space and the space of fields (ϕ) which it parametrizes are intimately connected in quantum field theory. This is especially clear if the effective field theory is a nonlinear model:

$$L_{\text{eff}} = \sigma_{ij}\gamma^{ab}\partial_a\phi^i\partial_b\phi^j = (g_{ij}\delta^{ab} + e_{ij}\varepsilon^{ab})\,\partial_a\phi^i\partial_b\phi^j \tag{7}$$

The "transport tensor" σ_{ij} splits naturally into a symmetric piece g_{ij} (in the isotropic case this tensor reduces to $g_{ij} = \sigma_{xx}\delta_{ij}$), which is the target space metric of the nonlinear model, and an antisymmetric piece $e_{ij} = \sigma_{xy}\varepsilon_{ij}$ ("the Hall conductivity"), which is a topological invariant of the nonlinear model, called torsion.

Our question can now be reformulated: what is the target space whose isotropic metric combines with torsion to give a modular invariant complex structure? The answer is the torus. Furthermore, the general toroidal model is equipped with *two* natural complex structures:

$$\begin{aligned}\tau &= [s_{xy} + i\,\mathrm{vol}(g)]/\sigma_{yy} \\ \sigma &= \sigma_{xy} + i\,\mathrm{vol}(g)\end{aligned} \tag{8}$$

where the volume of the metric is $\mathrm{vol}(g) = (\sigma_{xx}\sigma_{yy} - s_{xy}^2)^{1/2}$, τ is the usual complex structure (modular) parameter of the torus (which determines its shape), and σ is the complexified Kähler form (which determines its size and torsion). The real part of the Kähler form encodes topological (instanton) contributions to the model, which do not appear in conventional algebraic geometry, and therefore are of central interest in attempts to quantize gravity (string theory).

Setting $s_{xy} = 0$, we can write this parametrization as

$$\begin{aligned}\tau &= i\eta^{-1} \\ \sigma &= \sigma_{xy} + i\eta\sigma_{xx}\end{aligned} \tag{9}$$

which exhibits rather clearly the connection with the isotropic case ($\eta = 1$).

While the virtues of this model are obvious (it is modular invariant and is in no way restricted to the isotropic case), it is clear from geometry alone that the toroidal sigma model (the linear model) cannot represent the low-energy degrees of freedom of the Hall system. First, the topological term (the "torsion"), which would encode the degrees of freedom associated with the transverse (Hall) conductivity, is nontrivial iff the world-sheet (the sample) topology is nontrivial, in obvious contradiction with experiment. Second, it is always Gaussian, i.e., always critical (conformal) for all values of all parameters, while the Hall system is only in possession of isolated critical points.

We must therefore extend our search to a larger class of models, where the virtues but not the vices of the toroidal model are maintained. They can *not* (generically) be sigma models, since this leads unavoidably to the linear model described above.

Fortunately, there is a rather universal representation of two-dimensional models which is perfectly suited for this task, namely the Coulomb gas (CG) picture. Most, if not all, conformal field theories and statistical models can be mapped onto this system, and there is a simple modular invariant family whose basic degrees of freedom seem to correspond rather closely to the anyonic quasiparticles believed to be responsible for the quantum Hall effect. Remarkably, it contains the toroidal model in a special continuum limit.

The form of these models is easy to motivate in the anyonic quasiparticle picture. Although we are trying to solve a conductivity problem, in two dimensions this is equivalent to an electrostatics problem. Using the latter language, which is customary in the CG representation of ("modified Gaussian") conformal field theories, we build the model from "magnetic" (m) and "electric" (n) degrees of freedom.

In the general anisotropic case the "charges" (currents) $m_i(r)$ and $n_i(r)$ at position r (possibly on a lattice, if desired) carry a spatial index $i = x, y$, and the most general conventional Coulomb interaction is

$$H_C = \int dr\, dr' \, \{n^i(r)\, \varepsilon_{ij}^{-1} n^j(r') + m^i(r)\, \mu_{ij}^{-1} m^j(r')\}\, G(r, r') \tag{10}$$

where the Coulomb Green function in two dimensions is $G(r, r') = \langle r| \nabla^{-2} |r'\rangle = -(2\pi)^{-1} \log |r - r'|$, and ε and μ are the electric and magnetic permittivities. In a no-loss medium they are real and symmetric, and they should satisfy the vacuum dispersion relation $\varepsilon\mu = 1$ since they correspond to the (dissipative) conductivity and resistivity, which by definition are inverse tensors. Alternatively, since the model in the continuum limit becomes a sigma model, we can also regard ε_{ij} as a target space metric g_{ij} and μ_{ij} as its inverse g_{ij}^{-1}. Notice that the hydrodynamic vortex interaction is also of this form, which fits well with the interpretation of quasiparticles as topological excitations of the two-dimensional "Hall liquid." Note also that the charges can be regarded as the charges of a pair of coupled Z_p-symmetric spin models on dual lattices, which is where Cardy[13] first discovered the symmetries discussed here. The "percolation" model obtained by analytic continuation in p to $p \to 1$ appeared in the isotropic case as a candidate for the replica limit of a mesoscopic localization theory of anyons.[6] In the opposite continuum ("infinite replica") limit ($p \to \infty$) it becomes the toroidal sigma model. The main criteria for focusing on this class of models is that they are modular invariant in the isotropic case and automatically include the anisotropic generalization. They may turn out to be the only such models in two dimensions, but we shall not pursue this mathematical question here.

The physical interpretation of H_C in the quantum Hall system is that it represents the static properties of the quasiparticles and quasiholes which can be excited from a given ground state (labeled by a Hall fraction), provided that we can also encode their anyonic properties in a natural way. It is well known how to do this: add an Aharanov–Bohm-type interaction

$$H_{AB} = 2i \int dr\, dr' \, m^i(r)\, \Theta_{ij}(r - r')\, n^j(r') \tag{11}$$

and recall that the "magnetic monopoles" (vortices) pick up an electric charge proportional to a new (topological) "theta parameter," so that the electric current is modified:

$$n_i(r) \to n_i(r) + \frac{\theta}{2\pi} m_i(r) \tag{12}$$

$\Theta_{ij}(r)$ has a complicated angular dependence in general, but is an antisymmetric matrix which is symmetric in the argument r. In two dimensions it must be related to the argument of the complex Coulomb interaction $\mathrm{Log}(r-r') = \log|r-r'| + i\arg(r-r')$, but I have not sorted this out. The Hall conductivity is obviously to be identified with the angle $\theta/2\pi$.

The CG Hamiltonian $H_{\mathrm{CG}} = H_{\mathrm{AB}} + H_{\mathrm{C}}$ depends on four real parameters and the corresponding partition function is invariant under two copies of the modular group acting *linearly* on two unique complex combinations of these.

Before showing this, let us try to identify the *physical origin* of these remarkable symmetries. H_{CG} is invariant under "translations" T: $\theta \to \theta + 2\pi$ accompanied by a redefinition of the electric current $n \to n - m$. It also has built in a "duality" invariance S: $g \to g^{-1}$, which is familiar from many areas of physics, and which appears whenever topologically distinct degrees of freedom appear symmetrically (i.e., on equal footing) in the theory. Examples include Maxwell's equations in the presence of magnetic monopoles, spin-wave–vortex (order–disorder) duality in spin models, "$R \to 1/R$" duality in conformal field theory (string theory), etc. Since S and T do not commute, they generate an infinite discrete group of symmetries (the modular group $SL(2, Z)$) acting on the parameter space of the model, i.e., the partition function.[3] *In short, if the physics of the model is periodic in one parameter, a topological angle say, and self-dual in another, the coupling constant, say, which separately are two rather innocuous Abelian symmetries, then they may conspire to generate an infinite, non-Abelian discrete group. When this happens the parameter space geometry is so constrained that the phase and RG flow diagram is all but fixed.*

In order to make the symmetries manifest and show how the parameters conspire to become two independent complex structures on parameter space, we rotate H_{CG} to a complex basis. The basis is chosen so that the theory is holomorphically factorized at conformal fixed points, which is achieved by rotating every vector index with

$$R = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{13}$$

[3] Note that if for some reason the period of the angular (topological) dependence is changed, to T^2, say, then the model will only be invariant under a congruence subgroup of the modular group.

so that $(m_x, m_y) \to (m, \bar{m}) = (m_x - im_y, m_x + im_y)$, etc. After a lot of tedious algebra it then transpires that [the superscript s denotes the symmetric (dissipative) part]

$$
\begin{aligned}
H_C = & -2\pi \int dr\, dr' \, \{m^i(r)\, \sigma^s_{ij} m^j(r') \\
& + [n^i(r) + \sigma_{xy} m^i(r)]\, \rho^s_{ij} [n^j(r') + \sigma_{xy} m^j(r')]\}\, G(r, r') \\
= & \int dr\, dr' \, \{(\tau\bar{\tau} + 1)[J(\sigma, \tau)\, \bar{J}(\bar{\sigma}, r') + J(\bar{\sigma}, r)\, \bar{J}(\sigma, r')] \\
& + (\tau + i)(\bar{\tau} + i)[J(\sigma, r)\, J(\sigma, r') + J(\bar{\sigma}, r)\, J(\bar{\sigma}, r')] \\
& + (\tau - i)(\bar{\tau} - i)[\bar{J}(\sigma, r)\, \bar{J}(\sigma, r') + \bar{J}(\bar{\sigma}, r)\, \bar{J}(\bar{\sigma}, r')]\} \frac{\log |r - r'|}{\operatorname{Im} \tau \operatorname{Im} \sigma}
\end{aligned} \tag{14}
$$

where the complex parameters σ and τ are precisely those of the toroidal model [see Eq. (8)] and the currents are

$$
\begin{aligned}
J(\sigma, r) &= n(r) + \sigma m(r); \qquad & \bar{J}(\bar{\sigma}, r) &= n(r) + \bar{\sigma} m(r) \\
\bar{J}(\sigma, r) &= \bar{n}(r) + \sigma \bar{m}(r); \qquad & \bar{J}(\bar{\sigma}, r) &= \bar{n}(r) + \bar{\sigma} \bar{m}(r)
\end{aligned} \tag{15}
$$

The peculiar looking denominator in H_C is precisely what is needed to make H_{CG} invariant[4] under S transformations, as is easily verified in the isotropic case:

$$
\begin{aligned}
\sigma &\to -1/\sigma; \qquad & n &\to -m; \qquad & m &\to m \\
\tau &\to -1/\tau; \qquad & n &\to in; \qquad & m &\to im
\end{aligned} \tag{16}
$$

The physical meaning of $S(\tau)$ is simple: it is just the statement that the physics must be unchanged if the x and y axes are interchanged. No correspondingly simple interpretation of $S(\sigma)$ is available. This is in fact precisely where all the subtlety of the quantum Hall transport is buried. This symmetry has no classical analogy, unlike $S(\tau)$, which is well known in algebraic geometry, and is generated by nonperturbative effects in the strongly correlated electron gas. A similar symmetry was first discovered in a sigma-model description of certain string models, where it is generated by the instanton corrections which appear in the model at short scales. The elucidation of the symmetries of this "complexified Kähler cone" is the central nonperturbative problem in the quantum Hall effect, since it

[4] H_{AB} should perhaps be absorbed in H_C by converting $\log |r - r'|$ to $\operatorname{Log}(r - r')$, so that the symmetries of H_{CG} would be even more manifest.

governs the phase and RG flow diagram in the space of physically relevant transport parameters.

It is also easily seen that $T(\sigma)$ is a symmetry

$$\sigma \to \sigma + 1; \qquad n \to n - m; \qquad m \to n \tag{17}$$

but there is no corresponding symmetry involving τ. This means that it is not quite the toroidal model which appears in the continuum limit where we let the number of spin components p go to infinity, but rather the "covering" of the torus which tesselates the upper half-plane $H(\tau)$ (i.e., τ parametrizes the Teichmüller space of the torus).

In summary, while the physics of these models changes dramatically as a function of the discrete label p, the complex structures and symmetries of the parameter space do not.

It was argued in ref. 6 that the model encoding the universal conductivity properties of the isotropic Hall system is obtained by analytic continuation in p to the value $p = 1$. The physics of this is that of two coupled percolation models, which allowed us to deduce that the delocalization exponent is $\nu = 4/3$, or $7/3$ if tunneling corrections should be taken into account, in apparent agreement with recent scaling experiments.

It would now be highly desirable if these scaling experiments could be repeated for anisotropic samples, since the geometry of the models discussed above has a remarkable experimental signature: *the location of all fixed points of the RG flow will be unchanged provided that the geometrical parametrization* (8) *is used.* In practice this means that if the currents are aligned with the xy coordinate system we need only rescale the isotropic dissipative coordinate σ_{xx} by η. For example, the integer delocalization fixed point controlling the transition between the integer Hall plateaus n and $n+1$, which can be located rather easily using temperature-driven flows,[4] is predicted to lie at $\sigma_{xy} = n + 1/2$ and $\eta\sigma_{xx} = \eta^{-1}\sigma_{yy} = (\sigma_{xx}\sigma_{yy})^{1/2} = 1/2$. In the isotropic case ($\eta = 1$) this agrees with the Princeton experiment.

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REFERENCES

1. R. E. Prange and S. M. Girvin, eds., *The Quantum Hall Effect*, 2nd ed. (Springer-Verlag, Berlin, 1990).
2. J. M. Leinaas and J. Myrheim, *Nuovo Cimento* **37**:1 (1977).

3. C. A. Lütken and G. G. Ross, *Phys. Rev. B* **45**:11837 (1992).
4. H. P. Wei, D. C. Tsui, and A. M. M. Pruisken, *Phys. Rev. B* **33**:1488 (1985); H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, *Phys. Rev. Lett.* **61**:1294 (1988); L. Engel, H. P. Wei, D. C. Tsui, and M. Shayegan, *Surf. Sci.* **229**:13 (1990).
5. S. Koch, R. J. Haug, K. v. Klitzing, and K. Ploog, *Phys. Rev. Lett.* **67**:883 (1991).
6. C. A. Lütken and G. G. Ross, *Phys. Rev. B* **48**:2500 (1993).
7. J. T. Chalker and P. D. Coddington, *J. Phys. C* **21**:2665 (1988).
8. B. Huckestein and B. Kramer, *Phys. Rev. Lett.* **64**:1437 (1990).
9. Y. Huo, R. E. Hetzel, and R. N. Bhatt, *Phys. Rev. Lett.* **70**:481 (1993).
10. S. Kivelson, D.-H. Lee, and S.-C. Zhang, *Phys. Rev. B* **46**:2223 (1992).
11. G. T. Einevoll and C. A. Lütken, *Anisotropic Quantum Hall Effect*, *Phys. Rev.*, to appear.
12. H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, K. W. West, and J. Spector, *Appl. Phys. Lett.* **58**:726 (1991).
13. J. Cardy, *Nucl. Phys. B* **205**:17 (1982).